Microscopic Description of Black Hole Entropy in String Theory

Taro Spirig

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Oxford University

Abstract

We review the first successful miscroscopic derivation of black hole entropy in string theory with the necessary background to understand the argument. In particular, we present a conformal field theory background and prove the Cardy formula. We also review certain aspects of superstring theory.

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Contents

1 Black Hole Thermodynamics

Black Holes are defined as regions which are causally disconnected from infinity. The boundary of the region is defined by a surface called the event horizon. There exist uniqueness theorems for black holes which in essence state that black holes are described by metrics of the Kerr family (see [1] or [2]). The mechanics of black holes can be reduced to four laws (see [3] or section 9 in [4]). The zero-th law of black hole mechanics states that the surface gravity, κ, which is the acceleration felt by an object at the horizon as measured from an observer at infinity, of a stationary black hole is constant over the event horizon. The first law states that the perturbation of a stationary black hole of mass M, area A, angular momentum J, angular velocity Ω_H and charge Q with electrostatic potential Φ must satisfy:

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J + \Phi\delta Q. \tag{1}$$

The second law states that the area A of a black hole, i.e. defined by its event horizon, does not decrease with time, $\delta A \geq 0$. Finally, the third law states in essence that $\kappa = 0$, i.e. an extremal black hole, cannot be realized in a finite time if the energy-momentum tensor T_{ab} of the perturbation satisfies the weak energy condition, i.e. $T_{ab}V^aV^b \geq 0$ for any causal vector V^a, close to the horizon (see [5], [6]).

As the laws of black hole mechanics are very similar to the laws of thermodynamics, it led to the idea that the second law corresponds to the second law of thermodynamics, which states that entropy is never decreasing. J. D. Bekenstein first conjectured with information theoretic arguments that a black hole must have an entropy S_{BH}, which is proportional to its area A, and that the generalized second law of thermodynamics is that the total entropy $S = S_{\text{BH}} + S_{\text{matter}}$ is non-decreasing, see [7]. This entropy formula derived from Gedanken experiments was confirmed by S. Hawking, when he derived by considering quantum field theory in curved spaces that a black hole is actually radiating particles as a black body at temperature $T = \kappa/2\pi$ (see [8], [4], or [28]). Now, as we know from thermodynamics that $TdS = dM$. We infer from equation (1) that the entropy of a black hole must be

$$S_{\text{BH}} = \frac{A}{4}, \tag{2}$$

the Bekenstein–Hawking entropy.

It would be now desirable to derive this entropy by counting microscopic states of the black hole as in statistical mechanics. However, if we consider a Schwarzshild black hole, the metric describing it is unique (see section 2.1 in [4]). Thus, when counting all possible geometric configurations, the entropy of a Schwarzshild black hole would be zero, which contradicts equation (2). This contradiction means that our theory is not suitable for such a computation (see [9]). On the other hand, a theory of quantum gravity should be able to describe the entropy microscopically, as black holes would arise as excitations of quantum mechanical states.

2 Conformal Field Theory and the Cardy Formula

We first review some facts about 2-dimensional conformal field theory before deriving the Cardy formula, which is central in the derivation of black hole entropy presented below. As this section is only meant as a review, we state those facts mostly without proofs. We follow for the most part [10] and [11] for the conformal field theory background. For the derivation of the Cardy formula, we follow [14] and [15].

2.1 Classical Conformal Field Theory

A conformal field theory is a field theory which is invariant under conformal transformations, that is transformations of the fields (i.e. not just diffeomorphism covariance) that leave the metric invariant up to some position-dependent factor (see section 2.4 in [18]). Conventionally, Euclidean coordinates are used, i.e. $(\sigma_0, \sigma_1) \equiv (\tilde{\sigma}_0, i\tilde{\sigma}_1)$ with $\tilde{\sigma}_0,\ \tilde{\sigma}_1 \in \mathbb{R}$, where σ_0 can be thought of the time coordinate and σ_1 is the space coordinate.
The stress-energy tensor is per definition given by:

$$T_{\alpha\beta} = \frac{-4\pi}{\sqrt{g}} \frac{\partial S}{\partial g^{\alpha\beta}}, \tag{3}$$

where S denotes the action of our theory and the normalization constant follows the string-theoretic convention. Its conservation is given by the equations $\nabla_\alpha T^{\alpha\beta} = 0$, where ∇ denotes the covariant derivative. For a conformally invariant theory, the action is obviously conformally invariant. We can compute the variation of the action with respect to a specific conformal transformation called Weyl transformation $\delta g_{\alpha\beta} = \epsilon g_{\alpha\beta}$ from which it follows that the stress-energy tensor is traceless, i.e. $T^\alpha_\alpha = 0$.
To proceed with quantization, we compactify the σ_1 direction to avoid any infrared divergence, i.e. $\sigma_1 \equiv \sigma_1 + 2\pi$. The conformal field theory is then defined on a cylinder. Additionally, we conformally transform our cylindrical coordinates to the flat complex plane where the tools of complex analysis can be used more easily, i.e. we define the complex coordinates $z = e^{\sigma_0 + i\sigma_1}$ and $\overline{z} = e^{\sigma_0 - i\sigma_1}$. We assume that z and $\overline{z}$ are independent while remembering that in fact $\overline{z} = z^*$. In these coordinates, we write $\partial \coloneqq \partial_z,\ \overline{\partial} \coloneqq \partial_{\overline{z}}$ and $g_{zz} = g_{\overline{z}\overline{z}} = 0,\ g_{z\overline{z}} = \frac{1}{2}$. The tracelessness of the stress-energy tensor becomes, $T_{z\overline{z}} = 0$. Using this fact, the conservation of the stress-energy tensor becomes $\partial T_{zz} = 0,\ \overline{\partial} T_{\overline{z}\overline{z}} = 0$. We can thus make the following definitions $T(z) \coloneqq T_{zz}(z) = T_{zz}(z, \overline{z}),\ \overline{T}(\overline{z}) = T_{\overline{z}\overline{z}}(\overline{z}) = T_{\overline{z}\overline{z}}(z, \overline{z})$. For the following discussion, we will only consider T and only state the results for $\overline{T}$ as the same discussion holds for $\overline{T}$.

2.2 Quantum Mechanical Conformal Field Theory

We note that with the coordinates we have defined above, the origin of the complex plane corresponds to the infinite past and each circle centered around the origin represents a constant time slice. Hence, dilations on the complex plane correspond to time evolution. This provides the intuition for radial quantization. In this formalism, we will consider all conformal operators on the complex plane to define our quantum mechanical conformal field theory, instead of usual fields from quantum field theory. We will denote arbitrary operators with $\mathcal{O}(z,\overline{z})$. From now on, conformal field theory (CFT) will refer to quantum mechanical conformal field theory.

2.3 OPE

The Operator Product Expansion (OPE) states that we can always write the product of two operators acting closely to one another inside time-ordered correlations functions as a weighted sum over all the other operators:

$$\langle \mathcal{O}_i(z,\overline{z})\mathcal{O}_j(w,\overline{w})\dots\rangle = \sum_k C_{ij}^k(z-w,\overline{z}-\overline{w})\,\langle \mathcal{O}_k(w,\overline{w})\dots\rangle\,,$$

where '...' denotes any other operator $\mathcal{O}_l(x,\overline{x})$ inserted at $|x-w|,|x-z|>|z-w|,\ |\overline{x}-\overline{w}|,|\overline{x}-\overline{z}|>|\overline{z}-\overline{w}|$, which is a necessary condition for the convergence of the OPE. By abuse of notation, we drop the time-ordered correlation symbols.

2.4 Central Charge

The central charge of a CFT tells us about the degrees of freedom in our theory. The central charge is defined by considering the OPE of the stress-energy tensor with itself (a motivation for this fact is given in [11], section 4.4):

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T, \tag{4}$$

where c is the central charge. Replacing all the variables with their complex conjugate, and T by $\overline{T}$ yields the definition of $\overline{c}$. From Noether's theorem, we know that the stress-energy tensor generates local conformal transformations, as it was found by a local variation of the action. The corresponding Noether charges are given by integrating the Noether current at a constant time slice, i.e. as a contour integral at constant radius on the complex plane:

$$Q = \frac{1}{2\pi i}\oint (dz\,T(z)\epsilon(z) + d\overline{z}\,\overline{T}(\overline{z})\overline{\epsilon}(\overline{z})), \tag{5}$$

for an infinitesimal local conformal transformation $\delta z = \epsilon(z)$ and $\delta\overline{z} = \overline{\epsilon}(\overline{z})$. The infinitesimal transformations property of any CFT operator is then generated by those charges as:

$$\delta_{\epsilon,\overline{\epsilon}}\mathcal{O}(w,\overline{w}) = [Q,\mathcal{O}]. \tag{6}$$

Thus, for smooth infinitesimal conformal transformations, which are non-singular at $w,\overline{w}$, we can Taylor expand $\epsilon(z) = \epsilon(w) + \partial\epsilon(w)(z-w) + \dots$. Assuming that $z,\overline{z}$ are independent, we find that for

$\delta_\epsilon T(w)$ the contour integral in equation (6) will pick up the following terms only:

$$\delta_\epsilon T(w) = \frac{1}{2\pi i}\oint dz\epsilon(z)[T(z), T(w)] = \epsilon(w)\partial T(w) + 2\partial\epsilon(w)T(w) + \frac{c}{12}\partial^3\epsilon(w),$$

where we used the Taylor expansion of ϵ, equation (4) and took care of the contour integrals as in [10] section 2.2. Integrating this equation out for a conformal transformation $z \to f(z)$, we find that the stress-energy tensor transforms as:

$$T(z) \to (\partial f)^2 T(f(z)) + \frac{c}{12}S(f,z); \quad S(f,z) = \frac{\partial f\partial^3 f - \frac{3}{2}(\partial^2 f)^2}{(\partial f)^2}$$

where S is called the Schwartzian. This leads to the Casimir energy which will be important when deriving the Cardy formula. Under the conformal transformation from cylindrical to flat complex coordinates, the stress-energy tensor transforms as: $T_{\text{cyl}}(\sigma_0 + i\sigma_1) = z^2T(z) - \frac{c}{24}$, since $(\partial f)^2 = \frac{1}{z^2}$ and $S(f,z) = \frac{1}{2z^2}$. Then the ground state energy on the cylinder is shifted by $E = -\frac{2\pi(c+\overline{c})}{24}$, assuming that the ground state energy on the plane is zero:

$$H := \int d\sigma T_{\text{cyl},\sigma_0\sigma_0} = \int d\sigma(T_{\text{cyl},u,u} + \overline{T}_{\text{cyl},\overline{u},\overline{u}}),$$

where $u = \sigma_0 + i\sigma_1$ and $\overline{u} = \sigma_0 - i\sigma_1$. Thus when going from the Hamiltonian on flat space to the Hamiltonian on the cylinder we will have to take into account this shift in energy.

2.5 Free Scalar Field

We now determine the central charge for the free scalar field. The action for the free scalar field is:

$$S = \frac{1}{2}\int dzd\overline{z}\partial_\alpha X\partial^\alpha X. \tag{7}$$

We now denote for convenience the tuples $(z,\overline{z})$ with the letter y. Using the path integral formalism, we compute:

$$0 = \int DX\frac{\delta}{\delta X(y)}[e^{-S}X(y')] = \int DXe^{-S}[\partial^2 X(y)X(y') + \delta(y-y')],$$

from which we infer that:

$$\langle\partial^2 X(y)X(y')\rangle = -\delta(y-y') \quad \text{and} \quad \langle X(y)X(y')\rangle = -\frac{1}{4\pi}\ln(y-y')^2,$$

using $\partial^2\ln(y-y')^2 = 4\pi\delta(y-y')$ (see section 4.3.1 in [11]). Using the equation of motion derived from the action, which is a free wave equation, we write the operators X in terms of left and right moving operators $X(y) = X(z) + \overline{X}(\overline{z})$. Considering only the right moving operators $X(z)$ as the discussion for the left moving operators is analogous, we find:

$$\langle X(z)X(w)\rangle = -\frac{1}{4\pi}\ln(z-w) \quad \text{and} \quad \langle \partial X(z)\partial X(w)\rangle = -\frac{1}{4\pi}\frac{1}{(z-w)^2}.$$

As we could have inserted any other operators in the time ordered product without changing the calculation, the OPE is hence:

$$\partial X(z)\partial X(w) = -\frac{1}{4\pi}\frac{1}{(z-w)^2} + \dots.$$

Classically, the stress-energy tensor is given by $T(z) = -2\pi\partial X\partial X$. Quantum mechanically, we need to normal order this product, as the two operators are acting at the same position which leads to divergences. We thus define:

$$T(z) \coloneqq -2\pi :\partial X\partial X: \; = -2\pi \lim_{z\to w}\left(\partial X(z)\partial X(w) - \langle \partial X(z)\partial X(w)\rangle\right),$$

which is finite per construction. Finally, we compute the OPE of two stress-energy tensors using Wick's contractions:

$$\begin{aligned} T(z)T(w) =& 4\pi^2 :\partial X(z)\partial X(z): \; :\partial X(w)\partial X(w): \\ =& 8\pi^2\left(\frac{1}{4\pi}\frac{1}{(z-w)^2}\right)^2 - 16\pi^2\left(\frac{1}{4\pi}\frac{:\partial X(z)\partial X(w):}{(z-w)^2}\right) = \frac{1/2}{(z-w)^4} + \dots, \end{aligned}$$

where we have taken into account all the possible contractions. This calculation implies that the central charge is $c = 1$ and similarly we would find that $\overline{c} = 1$.

2.6 Free Fermionic Field

We compute the central charge for free Majorana fermions. We only outline the proof since it follows from the same computation as for the free scalar field. The action is:

$$S[\psi] = \int dz d\overline{z}(\psi\overline{\partial}\psi + \overline{\psi}\partial\overline{\psi}), \tag{8}$$

where $\psi(z)$ is the chiral fermion and $\overline{\psi}(\overline{z})$ is the antichiral fermion. We can then compute the propagators (see [12], section 2.6 for more details) and find:

$$\psi(z)\psi(w) = \frac{1}{2\pi}\frac{1}{z-w}, \quad \overline{\psi}(\overline{z})\overline{\psi}(\overline{w}) = \frac{1}{2\pi}\frac{1}{\overline{z}-\overline{w}},$$

from which we also find:

$$\partial\psi(z)\partial\psi(w) = -\frac{1}{\pi}\frac{1}{(z-w)^3}, \quad \overline{\partial\psi}(\overline{z})\overline{\partial\psi}(\overline{w}) = -\frac{1}{\pi}\frac{1}{(\overline{z}-\overline{w})^3}.$$

The normal ordered stress-energy tensors are given by: $T(z) = -\pi : \psi\partial\psi :$, $\overline{T}(\overline{z}) = -\pi : \overline{\psi}\overline{\partial}\overline{\psi} :$. Finally, we make use of Wick's theorem and compute:

$$\begin{aligned} T(z)T(w) =& (-\pi)^2 : \psi\partial\psi : : \psi\partial\psi : \\ =& \pi^2 \langle \partial\psi(z)\psi(w)\rangle \langle \psi(z)\partial\psi(w)\rangle - \pi^2 \langle \psi(z)\psi(w)\rangle \langle \partial\psi(z)\partial\psi(w)\rangle + \ldots \\ =& \pi^2 \frac{1}{2\pi}\frac{(-1)}{(z-w)^2}\frac{1}{2\pi}\frac{1}{(z-w)^2} - \pi^2 \frac{1}{2\pi}\frac{1}{z-w}\frac{(-1)}{\pi}\frac{1}{(z-w)^3} + \ldots = \frac{1/4}{(z-w)^4} + \ldots, \end{aligned}$$

from which it follows that the central charge is $c = 1/2$ and similarly $\overline{c} = 1/2$.

2.7 Virasoro Algebra

We consider the Laurent series of the stress-energy tensors:

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n, \quad \overline{T}(\overline{z}) = \sum_{n\in\mathbb{Z}} \overline{z}^{-n-2} \overline{L}_n,$$

which we can invert, yielding:

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z), \quad \overline{L}_n = \oint \frac{d\overline{z}}{2\pi i} \overline{z}^{n+1} \overline{T}(\overline{z}).$$

It is interesting to note that these operators are in fact the generators from equation (5) for infinitesimal conformal transformation of the form $\epsilon(z) = z^{n+1}$ and $\overline{\epsilon}(\overline{z}) = \overline{z}^{n+1}$ respectively. Hence, rotations which are infinitesimally given by $\epsilon(z) = z$ and $\overline{\epsilon}(\overline{z}) = -\overline{z}$ (indeed, they are given by $z \to e^{ia}z \approx (1 + ia)z$ and $\overline{z} \to e^{-ia}\overline{z} \approx (1 - ia)\overline{z}$), are generated by $L_0 - \overline{L}_0$, while dilations, which are infinitesimally given by $\epsilon(z) = z$ and $\overline{\epsilon}(\overline{z}) = \overline{z}$, are given by $L_0 + \overline{L}_0$. Since dilations correspond to the Hamiltonian on the cylinder we find:

$$H = L_0 - \frac{c}{24} + \overline{L}_0 - \frac{\overline{c}}{24}, \tag{9}$$

where we have taken the Casimir energy into account. Finally, these charges form an algebra called the Virasoro algebra (see section 4.5.2 in [11] or section 3.3 in [10]):

$$[L_m, L_n] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \tag{10}$$

with the corresponding algebra for the $\overline{L}_n$'s and $\overline{c}$ as well.

2.8 Cardy Formula

We derive the Cardy formula which approximates the entropy of a 2-dimensional CFT in certain circumstances. To get a thermal partition function on the cylinder, we need to set the time coordinate to be periodic[1]. The cylinder hence becomes a torus and the corresponding complex plane is a lattice. Indeed, we consider the lattice defined by: $z \equiv z+1,\ z \equiv z+\tau,\ \mathrm{Im}(\tau) > 0$ where τ is called the modular parameter. We thus get a lattice of parallelograms which are parametrized by the modular parameter only. This parameter can be changed in certain ways such that the corresponding parallelogram is unaltered. In fact there are only two such transformations (and their combinations): the T-transformation which is given by $T : \tau \to \tau + 1$ and the S-transformation which is given by $S : \tau \to -\frac{1}{\tau}$. The first one indeed leaves the parallelogram unchanged as $z \to z + \tau + 1 = z + \tau$. The second one flips the two sides of the parallelogram yielding the original parallelogram up to some rescaling which is a conformal transformation.

We now derive the partition function on the torus in terms of the Virasoro operators found previously. We first note that for $\mathrm{Re}(\tau) = 0$ the parallelogram becomes a square. With the identification for the temperature $T = \mathrm{Im}(\tau)^{-1}$, we find the partition function:

$$Z = \mathrm{Tr}\left(e^{\beta H}\right) = \mathrm{Tr}\left(e^{-2\pi i \mathrm{Im}(\tau) H}\right) = \mathrm{Tr}\left(e^{-2\pi \mathrm{Im}(\tau)\left(L_0 + \overline{L}_0 - \frac{c+\overline{c}}{24}\right)}\right),$$

where the trace is over all states and we have used the identification of the Hamiltonian in terms of Virasoro operators as in equation (9). Now, with $\mathrm{Re}(\tau) \neq 0$ we get a parallelogram which is skewed. Hence, the rule $z \equiv z + \tau$ means that $\mathrm{Re}(z)$ will be identified with $\mathrm{Re}(z) + \mathrm{Re}(\tau)$. This has for effect that any point on the cylinder will be displaced by the operator $\exp\left(2\pi i \mathrm{Re}(\tau)\left(L_0 - \overline{L}_0\right)\right)$ as we know from our previous discussion that $L_0 - \overline{L}_0$ generates rotations at fixed time. Hence, taking into account this effect, we finally have the partition function:

$$Z(\tau, \overline{\tau}) = \mathrm{Tr} e^{-2\pi \mathrm{Im}(\tau)\left(L_0 + \overline{L}_0 - \frac{c+\overline{c}}{24}\right)} e^{2\pi i \mathrm{Re}(\tau)\left(L_0 - \overline{L}_0\right)} = \mathrm{Tr} e^{2\pi i \tau\left(L_0 - \frac{c}{24}\right)} e^{-2\pi i \overline{\tau}\left(\overline{L}_0 - \frac{\overline{c}}{24}\right)}, \tag{11}$$

where $\overline{\tau}$ is the complex conjugate of the modular parameter. The partition function can as well be written in the form of a sum over all eigenstates Δ and $\overline{\Delta}$ of the operators L_0 and $\overline{L}_0$ respectively:

$$Z(\tau, \overline{\tau}) = \sum_{\Delta, \overline{\Delta} \geq 0} \rho(\Delta, \overline{\Delta}) e^{2\pi i \tau\left(\Delta - \frac{c}{24}\right)} e^{-2\pi i \overline{\tau}\left(\overline{\Delta} - \frac{\overline{c}}{24}\right)} = \int_0^\infty d\Delta d\overline{\Delta}\ \rho(\Delta, \overline{\Delta}) e^{2\pi i \tau\left(\Delta - \frac{c}{24}\right)} e^{-2\pi i \overline{\tau}\left(\overline{\Delta} - \frac{\overline{c}}{24}\right)},$$

with $\rho(\Delta, \overline{\Delta})$ the number of states with eigenvalues Δ and $\overline{\Delta}$. We can then invert this equation by a contour integral going close to the real axis:

$$\rho(\Delta, \overline{\Delta}) = \int d\tau d\overline{\tau} Z(\tau, \overline{\tau}) e^{-2\pi i \tau\left(\Delta - \frac{c}{24}\right)} e^{2\pi i \overline{\tau}\left(\overline{\Delta} - \frac{\overline{c}}{24}\right)}, \tag{12}$$

which is hard to evaluate. Now, the crucial result by Cardy is that for certain properties of the CFT (assumed to be satisfied here), the partition function is modular invariant (see [13]). More specifically,

[1] The intuition behind the relation between periodic time and thermal system comes from quantum mechanics. For a time period of $i\beta = i/T$, the time evolution operator in quantum mechanics becomes $e^{iE(i\beta)} = e^{-E\beta}$ which is the Boltzmann factor for a system with temperature T [28].

it is invariant under S-transformations, i.e. $Z(\tau) = Z(-1/\tau)$. We now suppress the $\overline{\tau}$-dependency and restore it later, as the following computation holds for the $\overline{\tau}$-dependent part independently. Using modular invariance we find:

$$Z(\tau) = Z\left(-\frac{1}{\tau}\right) = e^{\frac{2\pi i}{\tau}\frac{c}{24}} Z_0\left(-\frac{1}{\tau}\right),$$

where $Z_0(-1/\tau) = \mathrm{Tr} e^{-\frac{2\pi i}{\tau}L_0}$. Equation (12) then becomes:

$$\rho(\Delta) = \int d\tau \exp\left[2\pi i\left(\tau\left(\Delta - \frac{c}{24}\right) - \frac{1}{\tau}\frac{c}{24}\right)\right] Z_0\left(-\frac{1}{\tau}\right) := \int d\tau e^{2\pi i f(\tau)} Z_0\left(-\frac{1}{\tau}\right). \tag{13}$$

We can then make a saddle point approximation to evaluate this integral. Indeed, this approximation states that the main contribution to the integral is given by the rapidly varying part of the integrand $e^{2\pi i f(\tau)}$ evaluated at its extremum (or saddle point), as long as $Z_0(-1/\tau)$ varies slowly around this extremal point. The extremum of f is given by:

$$\tau_0 = i\sqrt{\frac{c}{24(\Delta - c/24)}},$$

where $f'(\tau_0) = 0$. Now, we consider the large Δ limit, in which case τ_0 approaches zero, that is the high temperature limit (assuming τ is purely imaginary). We can then check if Z_0 is slowly varying around $\tau_0 \sim 0$, which would validate the saddle point approximation for large Δ. We write:

$$Z_0\left(-\frac{1}{\tau_0}\right) = Z_0\left(\frac{i}{\epsilon}\right) = \sum_{\Delta \geq 0} \rho(\Delta) e^{-\frac{2\pi\Delta}{\epsilon}},$$

where we have set $\tau_0 = i\epsilon$. Assuming that the lowest eigenvalue Δ_0 is zero, we infer that as $\epsilon \to 0$ only Δ_0 will contribute and thus $Z_0(-1/\tau_0)$ will be finite. The case $\Delta_0 \neq 0$ is discussed in [14]. In short, thanks to the modular invariance, we were able to control the high temperature behaviour of $Z(\tau)$ which in turn enabled us to use a saddle point approximation in the large Δ limit. We finally find:

$$\rho(\Delta) \approx \exp\big(2\pi i f(\tau_0)\big) = \exp\left(2\pi\sqrt{\frac{c}{6}\left(\Delta - \frac{c}{24}\right)}\right),$$

which counts the number of states with high energy Δ. In the microcanonical ensemble, the entropy is $S = \log\big(\rho(\Delta)\big)$ which yields (restoring the $\overline{\tau}$-part) the Cardy formula:

$$S = 2\pi\sqrt{\frac{c}{6}\left(\Delta - \frac{c}{24}\right)} + 2\pi\sqrt{\frac{\overline{c}}{6}\left(\overline{\Delta} - \frac{\overline{c}}{24}\right)}. \tag{14}$$

3 String Theory Background

String theory is the quantum mechanical study of interacting relativistic strings. Some fundamental fields such as the graviton are found in the spectrum of (super-)strings. Moreover, the usual divergences found in quantum theories of gravity are absent in string theory. It is therefore a good candidate for quantum gravity and thus it should give a complete description of black holes. We present here the string theoretic background necessary to understand the first microscopic derivation of the Bekenstein–Hawking Entropy formula. This background will be presented as a review of facts that are relevant for our purpose. We will try to direct the reader to literature where the derivations and more details can be found. We are largely following the discussions presented in [16] and [17]. We will only present free string theory assuming that spacetime is flat, i.e. we are describing strings at the lowest order in perturbation theory.

3.1 Bosonic String Theory

In this section, we review the most natural first step towards a description of quantum mechanical strings, i.e. the bosonic string theory.

3.1.1 Polyakov Action and Old Covariant Quantization

We describe a string embedded on a D-dimensional spacetime. The string defines a 2-dimensional worldsheet, which we parametrize with a time parameter τ and a spatial parameter σ that runs from $0 \leq \sigma \leq \pi$. Assuming that spacetime is flat Minkowski space, the action should minimize the area of this worldsheet, i.e. the Nambu–Goto action:

$$S = -T \int d^2\sigma \sqrt{G}; \quad G = \det(G_{\alpha\beta}), \quad G_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu, \tag{15}$$

where the coordinates are $\sigma^1 = \sigma,\ \sigma^2 = \tau$, X^μ is the μth component of the embedding of the string in spacetime, and T is the string tension defined as $T = (2\pi\alpha')^{-1}$ with the Regge slope parameter α' fixed to $\alpha' = \frac{1}{2}$. The Nambu–Goto action is however unpractical for quantization. We therefore introduce a string worldsheet metric $h^{\alpha\beta}$ as a zweibein in the action, i.e. the action should be equivalent to the Nambu–Goto action when imposing the equations of motion of $h^{\alpha\beta}$. As it can be inferred from equation (3), the zweibein constraint corresponds to the vanishing of the energy-momentum tensor $T_{\alpha\beta} = 0$. This process yields the Polyakov action:

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{-h} h^{\alpha\beta}(\sigma) \partial_\alpha X^\mu \partial_\beta X_\mu. \tag{16}$$

The Polyakov action is also called string sigma model action[2]. We can then fix two of the three symmetries of the Polyakov action by setting $h^{\alpha\beta}$ equal to the two-dimensional Minkowski metric $\eta^{\alpha\beta}$.

[2] A sigma model is in essence defined as a mapping from a configuration space (here the string worldsheet) to a target space (here the spacetime).

The remaining action is the one quantized in the old-covariant quantization scheme:

$$S = -\frac{T}{2}\int d^2\sigma \eta^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X. \tag{17}$$

This action corresponds exactly to the scalar field action from equation (7). Indeed, the residual symmetry is a conformal symmetry and thus the string fields X^μ define a CFT after quantization. The same analysis as in section 2.1 can be made when replacing z, $\overline{z}$ with the light-cone coordinates $\sigma_+ = \tau + \sigma$ and $\sigma_- = \tau - \sigma$. The zweibein condition thus boils down to the vanishing of T_{++} and T_{--}. To implement these conditions, we can also impose the vanishing of the Fourier components of T_{++} and T_{--} which we denote L_m and $\bar{L}_m$.
The Euler-Lagrange equation for the string fields X^μ is a free wave equation $(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0$. We need however to consider the boundary terms from the variation of the action:

$$-T\int d\tau \left[\partial_\sigma X_\mu \delta X^\mu|_{\sigma=\pi} - \partial_\sigma X_\mu \delta X^\mu|_{\sigma=0}\right]$$

which should vanish. There are two different strings to consider. It is either a closed string, i.e. $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\pi)$, in which case the boundary term obviously vanishes, or it is an open string. There are two consistent boundary conditions for the open string: Neumann boundary condition, $\partial_\sigma X_\mu|_{\sigma=0,\pi} = 0$, and Dirichlet boundary conditions, $X^\mu|_{\sigma=0} = X_0^\mu$, $X^\mu|_{\sigma=\pi} = X_\pi^\mu$ for some fixed X_0^μ, X_π^μ. We know that any solution to a wave equation can be written in an oscillator expansion. Additionally, we can separate the solution in left-moving and right-moving modes, i.e. $X^\mu(\tau,\sigma) = X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma)$ with:

$$X_R^\mu(\sigma_-) = \frac{x^\mu}{2} + \frac{l^2}{2}p^\mu\sigma_- + \frac{il}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-2in\sigma_-}, \quad X_L^\mu(\sigma_+) = \frac{x^\mu}{2} + \frac{l^2}{2}p^\mu\sigma_+ + \frac{il}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in\sigma_+},$$

where α_n^μ, $\tilde{\alpha}_n^\mu$ are Fourier components and l is the fundamental length of the string, that sets the scale of the theory. We will always set $l = 1$ for convenience. For the open string with Neumann boundary condition, the left- and right- moving modes are related to each other because of the boundary conditions. We therefore only find one set of oscillator modes α_n^μ:

$$X^\mu(\tau,\sigma) = x^\mu + l^2p^\mu\tau + il\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma). \tag{18}$$

We can then proceed with canonical quantization by imposing the following commutation relations in the open string case: $[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n}\eta^{\mu\nu}$, $[x^\mu, p^\nu] = i\eta^{\mu\nu}$, and additionally $[\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\delta_{m+n}\eta^{\mu\nu}$ for closed strings. These are almost the usual harmonic oscillator operators with $\alpha_m^\mu = \sqrt{m}\alpha_m^\mu$ and $\alpha_{-m}^\mu = \sqrt{m}\alpha_m^{\mu\dagger}$, $m > 0$. We then produce a Fock space by applying the operators α_{-m}^μ, $m > 0$ on a ground state $|0\rangle$, which is defined by $\alpha_m^\mu|0\rangle = 0$, $m > 0$. For the opens string, the mass squared of a state is:

$$M^2 = -2a + 2\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n.$$

where the second term is proportional to the number operator N and the first term is an additional normal ordering constant a. Similarly, for closed strings we have:

$$M^2 = -8a + 8\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = -8a + 8\sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n.$$

However, the Fock space has negative norm states which are called ghosts. Indeed, since for $\mu = \nu = 0$, $\eta^{\mu\nu} = -1$, the state $\alpha^0_{-m} |0\rangle$ has negative norm. All is not lost, as we still haven't set the L_m's and $\bar{L}_m$'s to zero. After quantization, these charges become operators which satisfy the Virasoro algebra from equation (10) with central charge $c = D$ (see section 2.2.2 in [17]). Quantum mechanically, it is not sensible to set the Virasoro operators to zero. The correct condition is instead that any physical state $|\phi\rangle$ should be annihilated by $L_m,\ m > 0$ and $(L_0 - a)|\phi\rangle = 0$, where $L_0 = p^2/2 + N$. With these constraints, the physical spectrum is free of ghosts when the spacetime dimension is $D = 26$ and $a = 1$ (see section 2.3.3 in [17]).

3.1.2 T-duality for Closed and Open Strings

A possible way to apply bosonic string theory to the 4-dimensional world we live in is to compactify 22 directions of our ($D = 26$)-dimensional bosonic string theory. This process is called Kaluza–Klein compactification. In this section, we describe Kaluza–Klein compactification on a circle and we describe an interesting duality between string theories on different circles: the T-duality.

Let us first discuss closed bosonic strings with one dimension, e.g. the 25th, compactified on a circle of radius R. Then the mode expansion of the X^μ, $\mu = 0, \ldots, 24$ does not change. In the 25th direction, the string can be winded up around the circle:

$$X^{25}(\sigma + \pi, \tau) = X^{25}(\sigma, \tau) + 2\pi R W,$$

where $W \in$ Z is called the winding number. Hence, we need to add a term $2RW\sigma$ to the mode expansion of $X^{25}(\sigma, \tau)$. Additionally, as the 25th dimension is compact, it follows that the momentum in this direction is quantized. Indeed, from quantum mechanics, the wave function can be written as a free wave $\exp(ip^{25}x^{25})$ which is invariant under the transformation $x^{25} \to x^{25} + 2\pi R$. Thus the momentum must be quantized such that $p^{25} = K/R$, where $K \in \mathbb{Z}$ is called Kaluza–Klein excitation number. Now the left and right movers are of the form:

$$X_R^{25}(\sigma_-) = \frac{1}{2}(x^{25} - \tilde{x}^{25}) + \sqrt{2\alpha'}\alpha_0^{25}\sigma_- + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{\mu}e^{-2in\sigma_-},$$

$$X_L^{25}(\sigma_+) = \frac{1}{2}(x^{25} + \tilde{x}^{25}) + \sqrt{2\alpha'}\tilde{\alpha}_0^{25}\sigma_+ + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^{\mu}e^{-2in\sigma_+},$$

where $\sqrt{2\alpha'}\alpha_0^{25} = \alpha'\frac{K}{R} - WR$, $\sqrt{2\alpha'}\tilde{\alpha}_0^{25} = \alpha'\frac{K}{R} + WR$ and $\tilde{x}^{25}$ is a constant which vanishes when taking $X = X_R + X_L$. The mass squared of the states in the non-compact space are then given by:

$$\frac{1}{2}\alpha' M^2 = (\alpha_0^{25})^2 + 2N_R - 2 = (\tilde{\alpha}_0^{25})^2 + 2N_L - 2,$$

from which it follows that $N_R - N_L = WK$, called level-matching condition, and

$$\alpha' M^2 = \alpha'\left(\left(\frac{K}{R}\right)^2 + (WR)^2\right) + 2N_R + 2N_L - 4.$$

Both formulas are invariant under the transformations: $W \to K$ and $R \to \tilde{R} = \frac{\alpha'}{R}$ and vice versa. This is the T-duality. The two bosonic string theories on a circle with radius R or $\tilde{R}$ and with momentum and winding numbers interchanged are actually equivalent. The other variables change as follows under T-duality $\alpha_0^{25} \to -\alpha_0^{25},\ \tilde{\alpha}_0^{25} \to \tilde{\alpha}_0^{25},\ X_R^{25} \to -X_R^{25},\ X_L^{25} \to X_L^{25}$. Let us now consider an open string compactified on the same circle. In this case, the string does not have any winding modes as it can always be contracted to a point. The momentum however is still quantized in the 25th direction, i.e. $p^{25} = K/R,\ K \in \mathbb{Z}$. We write X^{25} from equation (18) in terms of left and right movers, $X^{25} = X_R^{25} + X_L^{25}$:

$$X_R^{25}(\sigma_-) = \frac{x^{25} - \tilde{x}^{25}}{2} + \frac{1}{2}p^{25}\sigma_- + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}e^{-in\sigma_-},$$

$$X_L^{25}(\sigma_+) = \frac{x^{25} + \tilde{x}^{25}}{2} + \frac{1}{2}p^{25}\sigma_+ + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}e^{-in\sigma_+}.$$

As in the closed strings case, the T-duality maps $X_R^{25} \to -X_R^{25}$ and $X_L^{25} \to X_L^{25}$, then X^{25} will be mapped to:

$$\tilde{X}^{25} = X_L^{25} - X_R^{25} = \tilde{x}^{25} + p^{25}\sigma + \sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}e^{-in\sigma}\sin(n\sigma).$$

So the dual open string has no momentum as there are no term proportional to τ. Additionally, the dual string's ends are fixed in the 25th dimension as $\sin(\sigma)|_{\sigma=\{0,\pi\}} = 0$, i.e. $\tilde{X}(\tau, 0) = \tilde{x}$, $\tilde{X}(\tau, \pi) = \tilde{x} + \frac{\pi K}{R}$. Hence, under T-duality, the Neumann boundary conditions become Dirichlet boundary conditions and the momentum becomes a winding number. Since Dirichlet boundary conditions fixes the endpoints

of the string, it is sensible to have a winding number as the winded string cannot be contracted to a point without breaking.
Now, the endpoints of the strings can only move on a $(24+1)$-dimensional hyperplane. In fact, those hyperplanes are physical objects themselves, called D-branes (D for Dirichlet). Dp-branes are defined as being the $(p+1)$-dimensional hyperplane on which open strings can end[3]. Thus, we have just shown that a D24-brane appears naturally by T-dualizing an open string on a circle. As the open string with Neumann boundary conditions could freely move in all $(25+1)$ dimensions, it is actually ending on a D25-brane that wraps the circle of the 25th dimension. By repeatedly T-dualizing the open string in other directions, we can construct Dp-branes of any $p < 25$. In short, T-duality maps a D25-brane wrapping n circles to a D$(25-n)$-brane.

3.2 Supersymmetry

Bosonic string theory is unrealistic because, inter alia, as it does not realize fermionic degrees of freedom. To include fermions in string theory, it is required to use supersymmetry, which is a symmetry relating each bosonic degree of freedom to a corresponding fermionic one and vice versa. In this section, we discuss some aspects of supersymmetry without refering to string theory. We refer the reader to [21] and [22] for more details about supersymmetry.

3.2.1 Supersymmetry Algebra and Supermultiplets

A supersymmetry algebra is the fermionic extension of an algebra, such as the Poincaré algebra. We will in fact only consider super-Poincaré algebras in the following. It contains not only 'bosonic' generators denoted by the letter X but also 'fermionic' ones denoted by the letter Q, which are called supersymmetry generators. We denote with the letter $\mathcal{N}$ the number of supersymmetry generators. Together they satisfy an algebra of the form:

$$[X, X'] = X'', \quad [X, Q] = Q'', \quad \{Q, Q'\} = X,$$

the brackets $[\cdot,\cdot]$ denote commutators and $\{\cdot,\cdot\}$ anticommutators. Explicitly, the Poincaré algebra is given by translation generators P_μ as well as boosts and rotation generators $M_{\mu\nu}$. Additionally, the supersymmetry generators Q transform in some spinor representation ρ of the Lorentz group. The commutation relations are then (see section 3 of [21]):

$$[P_\mu, Q_a] = 0, \quad [M_{\mu\nu}, Q_a] = -\left(\mathcal{M}_{\mu\nu}\right)_a^{\ b} Q_b, \quad \{Q_a, Q_b\} = C^\mu_{ab} P_\mu + Z_{ab},$$

where a, b are spinor indices, $\mathcal{M}_{\mu\nu} = \rho(M_{\mu\nu})$, Z_{ab} is a central charge, meaning that it commutes with all other elements of the superalgebra, and C^μ_{ab} are some structure constants that can be fixed using consistency with Lorentz invariance and the super-Jacobi identity. The goal is then to find

[3] If we include background fields, Dp-branes are not necessarily hyperplanes, but can be curved.

a representation of this algebra which will then incorporate as well as relate bosonic and fermionic degrees of freedom.

A representation of the supersymmetry algebra is called a supermultiplet. Let us now motivate some general properties of supermultiplets by reviewing supersymmetric algebras in quantum mechanics, i.e. $D = 1$ and $\mathcal{N}$ arbitrary. In one dimension, the Poincaré algebra only contains time translations which are generated by the energy operator: $E = -i\frac{d}{dt}$. The supersymmetric algebra then reads for $I, J \in \{1, \ldots, \mathcal{N}\}$:

$$\{Q^I, Q^J\} = 2E\delta^{IJ} + Z^{IJ}, \quad [E, Q^I] = 0,$$

where Z^{IJ} is the central charge. We can then define the fermion number operator $(-1)^F$ which acts on bosonic states $|b\rangle$ as $(-1)^F |b\rangle = |b\rangle$ and on fermionic states $|f\rangle$ as $(-1)^F |f\rangle = -|f\rangle$. As the supercharges transform bosonic states in fermionic states and vice versa, it anticommutes with the fermionic number operator: $Q^I(-1)^F = -(-1)^F Q^I$. We then have:

$$\mathrm{Tr}\left((-1)^F\left(2E\delta^{IJ} + Z^{IJ}\right)\right) = \mathrm{Tr}\left((-1)^F\{Q^I, Q^J\}\right) = \mathrm{Tr}\left((-1)^F Q^I Q^J + Q^I(-1)^F Q^J\right) = 0,$$

where we have used the cyclicity of the trace. In particular, we have $0 = \mathrm{Tr}\left((-1)^F E\right) = \langle E\rangle\, \mathrm{Tr}\left((-1)^F\right)$. Hence, if the energy $\langle E\rangle$ of a supermultiplet is non-zero, the supermultiplet must contain as many fermionic and bosonic states such that $\mathrm{Tr}\left((-1)^F\right)$ vanishes. Thus, if we assume that the energy spectrum is discrete and that there is only a finite number of vacua, the Witten index, $I_W = \mathrm{Tr}\left((-1)^F e^{-\beta H}\right)$ will only get contributions from the vacua, i.e. $I_W = \mathrm{Tr}\left((-1)^F\right)|_{E=0} = n_B - n_F$ which is the difference between the number of bosonic and fermionic vacua. The Witten index is very important because it gives an information on the system which will not change when varying parameters. We have seen that there must be as many fermionic and bosonic degrees of freedom per multiplet. Hence, when varying the parameters of a supersymmetric theory, states can leave or hit the ground states but only in bosonic-fermionic pairs, which implies that the Witten index will remain constant. We will later use a very similar index which is also parameter invariant, the elliptic genus.

3.2.2 R-symmetry

An R-symmetry is an automorphism of the supersymmetry algebra, i.e. it leaves the supersymmetry algebra invariant. For a supersymmetry algebra with supersymmetry generators Q^I, $I = 1, \ldots \mathcal{N}$, an element R of an R-symmetry group will commute with the Poincaré algebra and act on the supersymmetry generators as:

$$[R, Q^I_a] = -\mathbf{R^I{}_J} Q^J_a,$$

where $\mathbf{R}$ is a representation of R.

As an example, we show that $D = 4$ $\mathcal{N} = 1$ supersymmetry algebra has the R-symmetry $U(1)$. In

4-dimensions, the maximal number of independent supersymmetries is $4\mathcal{N}$, which can be written in $\mathcal{N}$ complex Weyl spinors Q_α and their conjugate $\bar{Q}^{\dot{\alpha}}$. In the $\mathcal{N} = 1$ case, we thus have a complex supersymmetry generator Q_α and its conjugate $\bar{Q}^{\dot{\alpha}}$. An element q of $U(1)$ acts on the supersymmetry generators as:

$$Q_\alpha \to e^{-iq} Q_a, \quad \bar{Q}^{\dot{\alpha}} \to e^{iq} \bar{Q}^{\dot{\alpha}},$$

which clearly leaves the supersymmetry algebra invariant as these constants factor out of the (anti-) commutators. We can thus assign charges to the supersymmetry generators: -1 for Q_α and $+1$ for $\bar{Q}^{\dot{\alpha}}$, and let the R-symmetry elements R act as charge operators:

$$[R, Q_\alpha] = -Q_\alpha, \quad [R, \bar{Q}^{\dot{\alpha}}] = \bar{Q}^{\dot{\alpha}}.$$

Thus the supermultiplets will also have a quantum number $r \in \mathbb{R}$ corresponding to the R-symmetry charge and acting with Q_α will decrease r by one and acting with $\bar{Q}^{\dot{\alpha}}$ will increase it by one.

3.2.3 BPS States

Some massive multiplets called BPS states are particularly stable and they are therefore very useful. We motivate what BPS states are in the case of $\mathcal{N} = 2$, $D = 4$ supersymmetry.
For $\mathcal{N} = 2$, the supersymmetry generators are given by two complex Weyl spinors Q_α and $\bar{Q}_{\dot{\alpha}}$, which satisfy the algebra:

$$\{Q_1^I, \bar{Q}_{\dot{1}}^J\} = \{Q_2^I, \bar{Q}_{\dot{2}}^J\} = 2M\delta_J^I, \quad \{Q_1^I, Q_2^J\} = 2Z\epsilon^{IJ}, \quad \{\bar{Q}_{\dot{1}}^I, \bar{Q}_{\dot{2}}^J\} = 2\bar{Z}\epsilon_{IJ},$$

where $I, J = 1, \ldots, \mathcal{N}$ label the supersymmetry generators, ϵ_{IJ} is an anti-symmetric 2-tensor, Z is the central charge, and M is the mass of the state. We can then define the following operators:

$$a_1 = \frac{1}{\sqrt{2}} \left(Q_1^1 + \alpha_1 \bar{Q}_{\dot{2}2} \right), \ a_2 = \frac{1}{\sqrt{2}} \left(Q_2^1 - \alpha_2 \bar{Q}_{\dot{1}2} \right), \ a_\alpha^\dagger = (a_\alpha)^\dagger,$$
$$b_1 = \frac{1}{\sqrt{2}} \left(Q_1^1 - \alpha_1 \bar{Q}_{\dot{2}2} \right), \ b_2 = \frac{1}{\sqrt{2}} \left(Q_2^1 + \alpha_2 \bar{Q}_{\dot{1}2} \right), \ b_\alpha^\dagger = (b_\alpha)^\dagger,$$

where the α_1 and α_2 are pure phases. Choosing $\alpha_1 = \alpha_2 = e^{i\arg(Z)}$, these operators satisfy the following algebra:

$$\{a_\alpha, a_\beta^\dagger\} = (2M + |Z|)\delta_{\alpha\beta}, \quad \{b_\alpha, b_\beta^\dagger\} = (2M - |Z|)\delta_{\alpha\beta}. \tag{19}$$

Because of the positive definiteness of the Hilbert space, the right hand side of the anticommutators from equation (19) must be positive. In particular, we have $2M - |Z| \geq 0$. In the case $2M > |Z|$, we can then act with the 4 operators $a_\alpha^\dagger$ and $b_\beta^\dagger$ on a vacuum yielding 2^4 states. This is called a long multiplet. The extremal case $2M = |Z|$ is called the BPS bound, which is trivially realised for massless multiplets. When this bound is satisfied, the mutliplet gets shortened. Indeed, the algebra of the operators b vanishes. The operators $b_\alpha^\dagger$ acts as the zero operator on any state, as $\langle\Omega| b_\alpha b_\alpha^\dagger |\Omega\rangle = 0$ with Ω being the vacuum. Hence, when the BPS condition is satisfied, only half of the supersymmetry is realised. For $\mathcal{N} > 2$, the central charge matrix is antisymmetric. It can be transformed in a block diagonal form with the rth block of the form $Z^{IJ} = 2\epsilon^{IJ} Z$ which corresponds to the $\mathcal{N} = 2$ central charge structure. We could thus repeat the same process outlined for the case $\mathcal{N} = 2$ and we would get for $\mathcal{N}$ even, $\mathcal{N}/2$ pairs of operators satisfying the algebra of equation (19). When k of these $\mathcal{N} = 2$ algebras satisfy the BPS bound, we get $k/\mathcal{N}$ supersymmetry preserving multiplets. For the maximal $k = \mathcal{N}/2$, we call such states half-BPS states.

Now, BPS states are stable, because their masses are minimal with respect to the central charges of the theory and thus the BPS states cannot decay into states with smaller masses. Hence, if we change some parameters of a system without breaking the supersymmetry, the BPS states will remain unchanged, with the only caveat being that another representation could become degenerate with the BPS multiplet.

Finally, we note that this discussion can be generalized to extended objects in higher dimensions such as the D-branes we already discussed.

3.2.4 Superspace

Ultimately we are interested in constructing supersymmetric field theories. The most straightforward way to construct a supersymmetric field theory is to start with a field $\phi(x)$ which commutes with the complex conjugate supersymmetry generators $\bar{Q}_{\dot{\alpha}}$, that is the analogue of the ground state in our previous discussion of supermultiplets. Then one can construct other fields by acting on ϕ with the supersymmetry generator Q_α until the representation closes, i.e. only derivatives of the previously found fields are produced when acting again with the supersymmetry generator (see section 3.3 [22]). This yields a supermultiplet of fields and one can construct a supersymmetric Lagrangian out of those fields. This procedure is however quite complicated. For $\mathcal{N} < 4$, there exists a formalism which makes supersymmetric field theories much more natural. As Poincaré-symmetric field theories are easily defined on Minkowski space, supersymmetric field theories could be more naturally defined on an extension of spacetime, i.e. an extension which includes spacetime symmetries generated by the supersymmetry generators. Such an extension is called superspace. For example, for the $\mathcal{N} = 1, D = 4$ case, superspace is constructed by adding 4 Grassmanian coordinates $\theta^\alpha, \bar{\theta}_{\dot{\alpha}}$ to the usual spacetime coordinate x^μ. Then supersymmetry generators are viewed as translations along these "fermionic" directions. More details on the construction of superspace can be found in section 4.2 of [21] and section 4 of [22].

3.3 Superstring Theory

We will now discuss how to make the action of string theory supersymmetric. They are mainly two equivalent formalism to achieve this. The first one is the Ramond–Neveu–Schwarz (RNS) formalism which imposes supersymmetry on the string worldsheet. The second one is the Green–Schwarz (GS) formalism which is supersymmetric in Minkowski spacetime or even more general spacetimes. Both formalism have their advantages and disadvantages. We will first present the GS formalism because it makes supersymmetry more visible. We then discuss very briefly the RNS formalism. As in bosonic string theory where we assumed that $D = 26$, here we will assume that spacetime is 10-dimensional, which appears, inter alia, as a requirement for the absence of ghosts in the physical spectrum of the RNS formalism. For more details on both formalism, see sections 4 and 5 of [16] or [17].

3.3.1 GS Formalism

As mentionned above we will want to make spacetime supersymmetric, which is most naturally done by extending the 10-dimensional Minkowski space to superspace. We therefore map the string worldsheet to superspace with $X^\mu(\sigma,\tau)$ as before and introducing some additional fermionic maps $\Theta^a(\sigma,\tau)$, where a denotes the spacetime spinor index. We are actually interested in the $\mathcal{N} = 2$ case which is called type II superstring theory. Now, $\mathcal{N} = 2$ means that we have to introduce two fermionic coordinates Θ_1, Θ_2 and since we are in 10-dimensions we can choose them to be Majorana-Weyl spinors. There are two possibilities for the chiralities of these fermionic coordinates. They could either have different chiralities or have the same chirality. The first case is called type IIA theory and the second one type IIB. In terms of the chirality operator $\Gamma_{11} = \Gamma_0\Gamma_1 \dots \Gamma_9$, where $\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}$ is the Dirac algebra in 10-dimensions, we have $\Gamma_{11}\Theta^A = (-1)^{A+1}\Theta^A$ for type IIA, and $\Gamma_{11}\Theta^A = \Theta^A$ for type IIB, with $A = 1, 2$. Then supersymmetry in superspace can be defined as infinitesimal transformations of the superspace coordinates: $\delta\Theta^{Aa} = \epsilon^{Aa}$, $\delta X^\mu = \bar{\epsilon}^A\Gamma^\mu\Theta^A$. With this definition, we indeed have the structure of a super-Poincaré algebra since the commutator of two infinitesimal supersymmetry transformations give $[\delta_1, \delta_2]\Theta^A = 0$ and $[\delta_1, \delta_2]X^\mu = -2\bar{\epsilon}_1^A\Gamma^\mu\epsilon_2^A$, which is a translation of the X^μ coordinate, i.e. a Poincaré transformation (see section 5.1 in [16]). We can now define the combination $\Pi^\mu_\alpha = \partial_\alpha X^\mu - \bar{\Theta}^A\Gamma^\mu\partial_\alpha\Theta^A$, where $\alpha = 1, 2$ for a string, which is invariant under the supersymmetry transformations:

$$\delta\left(\partial_\alpha X^\mu - \bar{\Theta}^A\Gamma^\mu\partial_\alpha\Theta^A\right) = \partial_\alpha\left(\bar{\epsilon}^A\Gamma^\mu\Theta^A\right) - \bar{\epsilon}^A\Gamma^\mu\partial_\alpha\Theta^A - \bar{\Theta}^A\Gamma^\mu\partial_\alpha\epsilon^A = 0,$$

where we have used the fact that $\partial_\alpha\epsilon^A = 0$ as it is a global transformation. This calculation is a good example of the convenience of superspace. With this supersymmetric extension of X^μ it would be most natural to simply replace X^μ by Π^μ in the Nambu-Goto action from equation (15). However, this action has too many fermionic degrees of freedom. In fact, we only need half of them as can be infered by analyzing the equations of motion of the action which we denote S_1. This is related to a gauge symmetry called κ-symmetry. If this symmetry holds, half of the fermionic degrees of freedom can be gauged away. By imposing κ-symmetry, we must introduce a second term S_2 of the Chern-Simons type in the action (see section 5.1 and 5.2 of [16]). Introducing again a string worldsheet $h_{\alpha\beta}$

as a zweibein we finally have:

$$S = S_1 + S_2,\ S_1 = -\frac{1}{2\pi}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\Pi_\alpha\cdot\Pi_\beta,$$
$$S_2 = \frac{1}{\pi}\int d^2\sigma\epsilon^{\alpha\beta}\left[-\partial_\alpha X^\mu\left(\bar{\Theta}^1\Gamma_\mu\partial_\beta\Theta^1 - \bar{\Theta}^2\Gamma_\mu\partial_\beta\Theta^2\right) - \bar{\Theta}^1\Gamma^\mu\partial_\alpha\Theta^1\bar{\Theta}^2\Gamma_\mu\partial_\beta\Theta^2\right].$$

Now, for quantizing this action, we use light-cone gauge quantization, because it simplifies drastically the equations of motions which are non-linear in X^μ and Θ^a. Light-cone gauge consists in fixing all the symmetries of the action before quantizing. As for the Polyakov action from equation (17), we can fix two out of the three symmetries of the action by setting $h^{\alpha\beta} = e^\phi\eta_{\alpha\beta}$, with e^ϕ chosen for later convenience. The remaining symmetry is superconformal symmetry. In light-cone gauge, we choose to fix this remaining symmetry as well before quantizing. The downside of this method is that the formalism becomes non-covariant but it is still sufficient to find the spectrum of the theory, which is our main interest. To fix this symmetry we rewrite the coordinates in light-cone coordinates $X^+ = (X^0 + X^{D-1})/\sqrt{2}$ and set all the oscillators in this direction to zero, i.e. $X^+ = x^+ + p^+\tau$, which leaves only 8 independent degrees of freedom. The corresponding gauge choice for the fermionic coordinates is $\Gamma^+\Theta^A = 0$, where $\Gamma^\pm = (\Gamma^0 \pm \Gamma^9)/\sqrt{2}$. Imposing this gauge fixing, the equations of motion become linear. In particular, we get two wave equations propagating in opposite directions for the fermionic coordinates: $(\partial_\tau + \partial_\sigma)\Theta^1 = 0$, $(\partial_\tau - \partial_\sigma)\Theta^2 = 0$, which follows from the relative minus sign between the two coordinates in S_2. Light-cone gauge implies that only X^i, $i = 1,\ldots,8$ are independent degrees of freedom, which reduces the 9-dimensional Lorentz symmetry to a $SO(8)$ symmetry. Similarly, we have started with two Majorana–Weyl fermions Θ^A which in 10 dimensions have each 16 real components. In light-cone gauge, we only have 8 degrees of freedom for each Θ^A, which then transform under $Spin(8)$, the covering group of $SO(8)$. This group has two spinor representations with opposite chiralities namely $\mathbf{8_s}$ and $\mathbf{8_c}$. After light-cone gauge fixing, we can then identify for type IIA and IIB the remaining spinors components of $\sqrt{p^+}\Theta^A$ with $\mathcal{S}_{1,2}$, which transform in the $Spin(8)$ representations: for type IIA, $\mathbf{8_s} + \mathbf{8_c} = (\mathcal{S}_1^a, \mathcal{S}_2^{\dot{a}})$, and for type IIB, $\mathbf{8_s} + \mathbf{8_s} = (\mathcal{S}_1^a, \mathcal{S}_2^a)$, where $a, \dot{a} = 1,\ldots,8$. Since the equations of motion of the 8-dimensional spinors are wave equations travelling in opposite directions, we can write them in mode expansions, as in the bosonic string theory. Since we are only interested in type II theories, we only describe the closed string case (for open type I theory see [16]). In this case, we impose $\mathcal{S}^{Aa}(\sigma,\tau) = \mathcal{S}^{Aa}(\sigma+\pi,\tau)$ and we find two different sets of modes defined by:

$$\mathcal{S}^{1a} = \sum_{-\infty}^{\infty} S_n^a e^{-2in(\tau-\sigma)}, \quad \mathcal{S}^{2a} = \sum_{-\infty}^{\infty} \tilde{S}_n^a e^{-2in(\tau+\sigma)}.$$

We can then cannonically quantize the theory by imposing the anticommutation relations: $\{S^a_m, S^b_n\} = \delta_{m+n,0}\delta^{ab}$, $\{\tilde{S}^a_m, \tilde{S}^b_n\} = \delta_{m+n,0}\delta^{ab}$. The mass squared of states in the spectrum is then given by:

$$\alpha' M^2 = \sum_{n=1}^{\infty}\left(\alpha^i_{-n}\alpha^i_n + nS^a_{-n}S^a_n\right) = \sum_{n=1}^{\infty}\left(\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + n\tilde{S}^a_{-n}\tilde{S}^a_n\right), \tag{20}$$

where the normal ordering constants of the fermionic and bosonic modes exactly cancel. The operators S^a_0, $\tilde{S}^a_0$ commute with the mass operator and since both satisfy the Clifford algebra $\{S^a_0, S^b_0\} = \delta^{ab}$, $a, b = 1, \ldots, 8$ with the analog for $\tilde{S}^a_0$, the ground state is degenerate and lives in an irreducible representation of this algebra. The possible representations are a massless vector representation $\mathbf{8_v}$ and two massless spinor representations with opposite chiralities $\mathbf{8_c}$ and $\mathbf{8}_s$. Hence, each of the left- and right-movers have a degenerate ground state of the form $\mathbf{8_v} + \mathbf{8_c}$ or $\mathbf{8_v} + \mathbf{8_s}$. We thus have the ground state formed by $(\mathbf{8_v} + \mathbf{8_c}) \otimes (\mathbf{8_v} + \mathbf{8_s})$ for type IIA and $(\mathbf{8_v} + \mathbf{8_c}) \otimes (\mathbf{8_v} + \mathbf{8_c})$ for type IIB. We then have to write the irreducible representation content of these tensor products. We only describe the process for the type IIB theory as it is analogous for type IIA. We get the following decomposition: $\mathbf{8_v} \otimes \mathbf{8_v} = \mathbf{1} + \mathbf{28} + \mathbf{35}$, which corresponds to a scalar, the dilaton ϕ, an antisymmetric rank-two tensor, $B_{\mu\nu}$, and a symmetric traceless tensor, the graviton $G_{\mu\nu}$. This part of the field content is denoted NS-NS sector, which is common to both the type IIA and IIB theories. The rest of the decomposition is: $\mathbf{8_c} \otimes \mathbf{8_c} = \mathbf{1} + \mathbf{28} + \mathbf{35}_+$, which corresponds to a zero form C_0, a two-form potential C_2 and a four-form potential C_4 with a self-dual field strength $\tilde{F}_5$. This part of the field content is unique to type IIB superstring theory and is denoted R-R sector. The other possible tensor products yield the corresponding fermionic superpartners.

3.3.2 RNS Formalism

We discuss the RNS formalism, because we will later use explicitly an analog of the RR-sector of the closed superstring spectrum found in the RNS formalism.

In this formalism, we impose supersymmetry on the worldsheet. The procedure is very similar to the quantization of bosonic string theory only with additional fermionic degrees of freedom[4]. Indeed, we introduce two components spinors $\psi^\mu(\sigma, \tau)$ on the worldsheet. Writing each component of the spinor as ψ_+, ψ_-, we can infer that $\psi_\pm$ are Majorana–Weyl spinors. We can then write a supersymmetric action (for supersymmetry variation, see section 4.3 in [16]) by adding a fermionic part S_f to the bosonic string action from equation (17):

$$S_f = \frac{i}{\pi}\int d^2\sigma(\psi_-\partial_+\psi_- + \psi_+\partial_-\psi_+).$$

We note that this action corresponds exactly to the action from equation (8). Thus, X_μ and ψ_μ define two CFT's and with supersymmetry, the whole theory becomes a superconformal field theory (SCFT).

[4]As in the GS formalism, one can also use the superspace formalism, adding some fermionic coordinates on the worldsheet (see section 4.3 [16])

As the bosonic part of the action is exactly the same as in bosonic string theory, we can concentrate on the fermionic part. Taking the variation of S_f against ψ_+, ψ_-, we find the Dirac equations: $\partial_+\psi_- = 0$ and $\partial_-\psi_+ = 0$ which describes left and right moving waves. We also find boundary terms which must vanish:

$$\int d\tau(\psi_+\delta\psi_- - \psi_-\delta\psi_-)|_{\sigma=\pi} - (\psi_+\delta\psi_- - \psi_-\delta\psi_-)|_{\sigma=0} = 0.$$

In the closed string case, there are two possibilities to fulfill this condition given by $\psi_\pm(\sigma,\tau) = \pm\psi_\pm(\sigma+\pi,\tau)$. The two left- and right- moving $\psi_\pm$ are independent in the closed string case and can separately satisfy the periodic or antiperiodic condition. The case that will be used later is when both are periodic, which is called the Ramond–Ramond-sector (RR-sector). In this sector the mode expansions of the two fermions are:

$$\psi_-^\mu(\sigma,\tau) = \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2in(\tau-\sigma)}, \quad \psi_+^\mu(\sigma,\tau) = \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-2in(\tau+\sigma)}.$$

We can then proceed with canonical quantization of the bosonic modes $\alpha_n, \tilde{\alpha}_n$ as before and the fermionic ones by setting $\{d_n^\mu, d_m^\nu\} = \eta^{\mu\nu}\delta_{m+n,0}$, $\{\tilde{d}_n^\mu, \tilde{d}_m^\nu\} = \eta^{\mu\nu}\delta_{m+n,0}$.
As in the bosonic string theory we have to restrict the Fock space which can be found by acting with these operators. In the bosonic string theory, the restriction came from the zweibein constraint and gauge fixing of the conformal symmetry. Now that we have a supersymmetric theory, the conformal symmetry that lead to the Virasoro algebra becomes a superconformal symmetry with a super-Virasoro algebra (supersymmetric extension of the Virasoro algebra). Indeed, there exists a supercurrent G^α which is associated with local world-sheet supersymmetry. It has two components G_- and G_+ which correspond to left- and right- moving supercurrents, as from the conservation of the current we have $\partial_+G_- = 0$ and $\partial_-G_+ = 0$. We can then write a superconformal algebra with T_{++} and G_+ and another superconformal algebra out of T_{--} and G_-[5]. We thus infer that X_L, ψ_+, X_R, ψ_- define a $(\mathcal{N} = 1, \tilde{\mathcal{N}} = 1)$-superconformal field theory, where $\mathcal{N}$ and $\tilde{\mathcal{N}}$ denotes the numbers of supersymmetry generators for the left-movers algebra and for the right-movers respectively. For the remaining discussion we only consider the left-moving sector as the same discussion holds for the right-moving sector. The super-Virasoro algebra is defined by including the Fourier components G_m of the supercurrent:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{8}m^3\delta_{m+n,0}, \quad [L_m, G_n] = \left(\frac{m}{2} - n\right)G_{m+n},$$
$$\{G_m, G_n\} = 2L_{m+n} + \frac{D}{2}m^2\delta_{m+n,0}.$$

We note that L_0 is equal to $L_0 = \frac{1}{2}\alpha_0^2 + N$ where N is the number operator counting the number of bosonic and fermionic excitations. As the zweibein constraint consisted in the vanishing of the

[5]This follows from the fact that the OPEs of T_{++} with G_+ as well as T_{--} with G_- are closing, see section 10.1 of [18].

energy-momentum tensor, we impose similarly here the vanishing of the energy-momentum tensor as well as the supercurrent. Quantum mechanically, these constraints translate in: $L_m |\phi\rangle = 0,\ m > 0$, $G_n |\phi\rangle = 0,\ n \geq 0$ and $L_0 |\phi\rangle = 0$. With these conditions, the physical spectrum is free of ghosts in 10-dimensional spacetime. The physical spectrum is however not spacetime supersymmetric and so a projection, called GSO projection, needs to be used to constrain the physical spectrum again. After this projection, we actually find exactly the same spectrum as in the GS formalism.
Finally, we discuss the case of $(\mathcal{N} = 2, \tilde{\mathcal{N}} = 2)$-SCFT. In this case the same analysis can be followed with the only difference being that there are two supersymmetry generators $G_n^{\pm}$ for the left-moving sector and correspondingly $\tilde{G}_n^{\pm}$ for the right-moving sector. Moreover, there is a $U(1)$ R-symmetry which is generated by the operators $J_n,\ n \in \mathbb{Z}$, which commute with the superconformal generators of the left-moving sector as $[J_n, G_m^{\pm}] = \pm G_{m+n}^{\pm}$ (see section 2.4 in [35] for the complete algebra). We can thus associate to each state a number F_L, the fermion number, corresponding to the eigenvalue of the operator J_0. The superconformal generators G_m^+ raise its value by one and G_m^- lower it by one. For the right-moving sector, we denote the fermion number with F_R.

3.3.3 Half-BPS Branes

The n-forms with $n = 0, 2, 4$ found in the type IIB superstring theory couple to extended objects of dimension $p = 1, 3, 5$. These objects indeed exist in our theory: they are the D-branes we have discussed in section 3.1.2. These physical objects are of particular importance in superstring theory. In this section, we discuss some properties of D-branes, in particular, the charges they carry and their stability. We then discuss T-duality for the type IIA and IIB theories.
D-branes couple to the n-forms of the type II theories as point particles couple to the Maxwell field in electrodynamics. Let us first recall the latter process. The Maxwell field is a one-form A with field strength tensor $F = dA$. The Hodge dual $\star$ maps n-forms to $(D - n)$-forms in D-dimensions. Assuming the existence of magnetic monopoles as well as electric ones, the Maxwell equations take the form: $dF = \star J_m,\ d \star F = J_e$, where J_e and J_m are the one-forms whose components are given by the respective charge densities and currents. For a point particle sitting at the origin, the charge densities are $\rho_e(x) = e\delta^3(x)$ and $\rho_m(x) = g\delta^3(x)$ with e the electric charge and g the magnetic one. To find the value of these charges, we can use Gauss law, i.e. we integrate Maxwell's equations over a ball surrounding the origin and then use Stokes theorem, which yields:

$$g = \int_{S^2} F, \quad e = \int_{S^2} \star F,$$

where S^2 is a 2-sphere centered at the origin. We can now generalize this discussion for n-form fields A_n with field strength tensor $F_{n+1} = dA_n$. The analog of Gauss law corresponds to integrating $\star F_{n+1}$ over a $(D-(n+1))$-sphere. As a $(D-p-2)$-sphere surrounds a p-dimensional object, we infer that A_n couples electrically with Dp-branes for which $p = n - 1$. The magnetic dual corresponds to integrating F_{n+1} over a $(n+1)$-sphere, which surrounds a Dp'-brane with $p' = D-3-n = D-p-4$. Hence, in $D = 10$ dimensions, an n-form field couples electrically to a D$(n-1)$-brane and magnetically to a D$(7-n)$-

brane. As in type IIB theory there are $0, 2, 4$-form fields, the Dp-branes with $p = -1, 1, 3, 5, 7$ have conserved charges which appear in the supersymmetry algebra as central charges. We will not consider the D(-1)-brane, which is localized in space and time, and its magnetic dual the D7-brane. Similarly, in type IIA, there are $1, 3$-form fields with corresponding charged Dp-branes for which $p = 0, 2, 4, 6$. Another convenient property of these D-branes is that they are half-BPS, as they preserve half of the spacetime supersymmetry. Indeed, for Dp-branes living in the $0, \ldots, p$-directions, we end up with a total supersymmetry generator Q given by (see section 6.2 in [16] and section 5.1 in [20]):

$$Q = Q_1 + \Gamma^{0 \ldots p} Q_2,$$

where Q_1 and Q_2 are the initial $\mathcal{N} = 2$ supersymmetry charges and Γ^i are Dirac matrices. This is indeed a valid supersymmetry generator as for type IIA, where p is even, $\Gamma^{0 \ldots p}$ changes the chirality of Q_2 such that it becomes the same as the chirality of Q_1 and for type IIB $\Gamma^{0 \ldots p}$ does not change the chirality of Q_2 as desired. Thus, there are only 16 of the initial 32 supersymmetries which are conserved. In short, the half-BPS D-branes are particularly stable as they satisfy the BPS bound.

3.3.4 Compactification and D-branes

We have shown in the previous section that Half-BPS D-branes are the sources of the gauge fields in type II superstring theories. Now, the electromagnetic field, i.e. a 1-form, couples to a particle, i.e. a D0-brane, in the following way:

$$S_{\text{int}} = e \int A = e \int d\tau A_\mu \frac{dx^\mu}{d\tau},$$

with e the electric charge. Similarly, a n-form couples to a Dp-brane with $p = n - 1$ as:

$$S_{\text{int}} = e_p \int A_{p+1} = e_p \int d^{p+1}\sigma A_{\mu_1 \ldots \mu_{p+1}} \frac{\partial X^{\mu_1}}{\partial \sigma_0} \cdots \frac{\partial X^{\mu_{p+1}}}{\partial \sigma^p},$$

where e_p is the charge of the brane and we have explicitly written the pullback to the worldvolume of the brane[6]. We now explore how the electric charge of a brane changes when we compactify along certain directions (see section 15.4 of [19]). Let us consider a Dp-brane which couples to a $(p+1)$-form A and wrap it around p circles in the $1, \ldots, p$ directions. Let $x^1, \ldots, x^p$ denote the compact directions with corresponding brane coordinates $X^1, \ldots, X^p$. In each of these compact directions, the brane can wind the corresponding circle W^k times, such that:

$$X^k(\tau, \sigma_1, \ldots, \sigma_k, \ldots, \sigma_p) = X^k(\tau, \sigma_1, \ldots, \sigma_k + \pi, \ldots, \sigma_p) + 2\pi W^k R^k, \quad k = 1, \ldots, p,$$

[6]Since a string defines a worldsheet, a brane defines a worldvolume.

where $0 \leq \sigma_k \leq \pi$ parametrizes the worldvolume of the brane and R^k is the radius of the circle in the kth-direction. Hence, as in the bosonic string theory, we must add the term $2R^kW^k\sigma_k$ to the brane coordinate X^k. Let us assume that $X^k(\tau, \sigma_1, \ldots, \sigma_p) = 2RW^k\sigma_k$ for $k = 1, \ldots, p$. In the non-compact world, the brane is now considered as a point with $X^m(\tau, \sigma_1, \ldots, \sigma_p) = x^m(\tau)$, where $m = 0, p+1, \ldots, 9$. The brane thus couples to the corresponding brane as:

$$S_{\text{int}} = \int A_{p+1} = \int d\tau d\sigma_1 \cdots d\sigma_p A_{\mu 12\ldots p} \frac{\partial X^\mu}{\partial \tau} 2W^1R^1 \cdots 2W^pR^p.$$

As the $(p+1)$-form is antisymmetric, μ can only refer to the non-compact dimensions. If we ignore the dependence of this form on the compact directions, we end up with:

$$S_{\text{int}} = W^1 \cdots W^k V_p \int A_{p+1} = \mu_p \int d\tau d\sigma_1 \cdots d\sigma_p \bar{A}_m(x(\tau)) \frac{\partial x^m}{\partial \tau},$$

where we defined $\bar{A}_m(x(\tau)) \coloneqq (\alpha')^{p/2} A_{m12\ldots p}(x(\tau))$ with $x(\tau)$ denoting all the non-compact directions and $V_p \coloneqq (2\pi R_1)(2\pi R_2)\cdots(2\pi R_p)$. We thus infer that the Dp-brane is identified as the source of a 1-form $\bar{A}$ in the non-compact world. Additionally, we can identify the $\bar{A}$-charge of the brane with its winding number in the compact directions. These facts hold more generally and will be crucial in the black hole entropy calculation.

When starting with the action of a superstring theory, compactification boils down to a very similar process. One must define fields such as $\bar{A}$ in the non-compact world from the original fields A of the theory (see section 2.2 of [23]). In general, this process is of course more involved and the geometry of the compact dimensions is very important. The D-branes in the original theory can then be wrapped along the compact directions, such that they couple to the fields of the non-compact world. For example, type IIB theory compactified over a 5-torus possess a 1-form gauge field in the non-compact world (see section 5.1 of [23]). If we then compactify a D1-brane of the original type IIB theory on one of the direction of the torus, the resulting D0-brane in the non-compact world will couple to the 1-form gauge field.

3.3.5 T-duality for Type II Superstring Theories

T-duality for type II string theories maps type IIA into type IIB and vice versa. We compactify the 9th-direction on a circle of radius R. From T-duality of closed strings (section 3.1.2), we know that T-duality will leave the left-movers invariant and flip the sign of the right movers, i.e. $X^9_L \to X^9_L$ and $X^9_R \to -X^9_R$. In the RNS formalism, it is easily seen that the corresponding fermionic degrees of freedom follow the same rule, i.e. the chirality of the fermionic right-movers is flipped while the chirality of the left movers is unchanged. In the GS formalism, T-duality acts as: $X^9_L \to X^9_L$, $X^9_R \to -X^9_R$, $S^a_1 \to S^a_1$, $S^{\dot{a}}_2 \to \Gamma^9_{b\dot{a}} S^b_2$, $S^a_2 \to \Gamma^9_{a\dot{b}} S^{\dot{b}}_2$, meaning that the right movers change chirality. We infer that T-duality maps type IIA string theory on type IIB and vice versa. The Dp-branes of the type IIA with p odd are mapped to the Dp'-branes of type IIB theory with p' even and vice versa. Indeed, from our discussion of T-duality for open strings, we know that a Dp-brane wrapped around a circle is mapped to a D$(p-1)$-brane under T-duality. So the half-BPS branes of one theory are mapped to the half-BPS states of the other theory.

3.3.6 Low-Energy Effective Actions

It is often easier to carry calculations in a low-energy limit. In string theory with flat background, this corresponds to the weak coupling limit $\alpha' \to 0$. In this regime, the massive states become extremely heavy as can be inferred from equation (20) and thus cannot be observed. We are thus interested in the massless modes. These can be described by a supergravity theory corresponding to the low-energy effective theory of the superstring theory. The type IIA supergravity can be derived by dimensional reduction from 11-dimensional supergravity. The type IIB supergravity needs to be constructed by imposing supersymmetry and gauge invariance. We already know the field content of type IIB supergravity from the derivation in section 3.3.1 of the field content of type IIB superstring theory. We should then write all the possible supersymmetric equations of motion and then write a corresponding action from which these equations could be derived. A problem in this procedure is that the self-duality condition of the field strength tensor $\tilde{F}_5 = \star\tilde{F}_5$ cannot be incorporated in the action. We thus write an action from which the correct equations of motion are derived when the self-duality constraint of the 5-form $\tilde{F}_5$ is additionally imposed. This action reads $S = S_{NS} + S_R + S_{CS}$ where:

$$
\begin{aligned}
S_{NS} &= \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right) \\
S_R &= -\frac{1}{4\kappa^2}\int d^{10}x\sqrt{-g}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right), \quad S_{CS} = -\frac{1}{4\kappa^2}\int C_4 \wedge H_3 \wedge F_3,
\end{aligned}
\tag{21}
$$

with $F_{n+1} = dC_n,\ H_3 = dB_2,\ \tilde{F}_3 = F_3 - C_0 H_3,\ \tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3$. The S_{NS} part of the action is constructed by NS-NS sector fields and is thus common to both type IIA and IIB theories. The S_R and S_{CS} parts of the action are built out of the R-R sector fields and is unique to type IIB theory. Another convention consists in replacing the metric with $\tilde{g} = e^{-\Phi/2}g$. This convention is called the Einstein-frame, whereas the convention used in equation (21) is the string-frame.

4 Microscopic Description of Black Hole Entropy

As string theory is a good candidate for quantum gravity, it should be able to describe black holes as quantum mechanical excitations of the fundamental objects in the theory. Indeed, D-branes at strong coupling produce black holes. One can compute the compactification of branes to see that the resulting metric and fields are that of a lower dimensional black hole. See section 2.3 of [25] for the example of a string compactified on a torus which yields an extremal black hole. As the degeneracy of the D-branes which produce a black hole can be calculated (at weak coupling), we will be able to determine the entropy of the black hole.

We will follow the first microscopic derivation of the Bekenstein–Hawking entropy formula by A. Strominger and C. Vafa [24]. The set-up corresponds to extremal black holes in type IIB string theory in five non-compact dimensions. The compact dimensions are defined by $K3 \times S^1$, where $K3$ is a Calabi–Yau 2-fold. As the geometry of the compact space will in fact not matter much in our discussion, we do not discuss compactification on Calabi–Yau manifolds (see section 9 of [16]).

4.1 The Bekenstein–Hawking Entropy

In this section, we compute the Bekenstein–Hawking entropy of a 5-dimensional black hole in type IIB supergravity. We show how the Reissner–Nordström metric which describes charged black holes arises. We are then able to compute the area of the black hole and thus the Bekenstein–Hawking entropy.

The low-energy effective action of type II string theory compactified on $K3 \times S^1$ in the Einstein frame is given by [24]:

$$S = \int d^5x\sqrt{-g}\left(R - \frac{4}{3}(\nabla\phi)^2 - \frac{1}{4}\exp\left(\frac{-4\phi}{3}\right)\tilde{H}^2 - \frac{1}{4}\exp\left(\frac{2\phi}{3}\right)F^2\right) = \int d^5x\mathcal{L}, \tag{22}$$

where $\tilde{H}$ is a 2-form field strength with one component tangent to the S^1, F is a RR 2-form field strength and R is the scalar curvature. The 2-form $\tilde{H}$ is derived from the NS-NS 3-form H_3 of equation (21) as $\tilde{H}_{\mu\nu} = (H_3)_{\mu\nu a}$ with the index a referring to the S^1 direction. The process of compactification of supergravity leads to enhanced supersymmetry and this action actually has $\mathcal{N} = 4$ supersymmetry. We consider a spherically symmetric 5-dimensional extremal black hole, whose event horizon describes a 3-sphere. The black hole couples to both fields F and $\tilde{H}$. As we have discussed in section 3.3.3, the charges of the black hole can be computed by Gauss' law, i.e.:

$$Q_H := \frac{1}{4\pi^2}\int_{S^3}\star\exp\left(\frac{-4\phi}{3}\right)\tilde{H}, \quad Q_F := \frac{1}{16\pi}\int_{S^3}\star\exp\left(\frac{2\phi}{3}\right)F, \tag{23}$$

where $\star$ denotes the Hodge dual which is required as we are integrating 2-forms on 3-spheres (recall that the hodge dual of a 2-form is a $(D-2)$-form so a 3-form in this case). The chosen convention ensures that Q_H and $\frac{1}{2}Q_F^2$ are integers. As the system we are considering is spherically symmetric, we can rewrite (23) for convenience in the following way:

$$2Q_H\epsilon_3 = \star\exp\left(\frac{-4\phi}{3}\right)\tilde{H}, \quad \frac{8Q_F}{\pi}\epsilon_3 = \star\exp\left(\frac{2\phi}{3}\right)F, \tag{24}$$

where ϵ_3 denotes the volume element on the unit 3-sphere S^3, i.e. $\epsilon_3 = \frac{1}{3!}\epsilon_{\mu\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho$, with x^μ the coordinate on the sphere.
Now, close to the horizon of the black hole the dilaton takes a constant value that we denote ϕ_h. As the equation of motion for ϕ is given by:

$$0 = \frac{d}{dt}\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} - \frac{\delta\mathcal{L}}{\delta\phi} \quad \Leftrightarrow \quad 0 = 16\nabla^2\phi + 2e^{-4\phi/3}\tilde{H}^2 - e^{2\phi/3}F^2, \tag{25}$$

we can find the value of $\phi = \phi_h$ by substituting equation (24) for $\tilde{H}$ and F:

$$8e^{4\phi_h/3}Q_H^2(\star\epsilon_3)^2 - e^{-2\phi_h/3}\frac{8^2Q_F^2}{\pi^2}(\star\epsilon_3)^2 = 0 \quad \Leftrightarrow \quad e^{2\phi_h} = \frac{1}{2}\left(\frac{4Q_F}{\pi Q_H}\right)^2. \tag{26}$$

Now, let us assume that the asymptotic value of the dilaton at infinity ϕ_∞ coincides with ϕ_h. We then note that for $\phi = \phi_h$, it follows from equation (25) that $2e^{-4\phi_h/3}\tilde{H}^2 = e^{2\phi_h/3}F^2$ and thus using equation (24) and (26) we find that

$$e^{-4\phi_h/3}\tilde{H}^2 + e^{2\phi_h/3}F^2 = 12\left(\frac{8Q_F^2Q_H}{\pi^2}\right)^{2/3}\epsilon_3^2.$$

Using this calculation, we can then compute the Einstein equations by varying the action with respect to the metric, i.e. it gives the Ricci curvature $R_{\mu\nu}$:

$$\begin{aligned} R_{\mu\nu} &= -\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left[\sqrt{-g}\left(-\frac{e^{-4\phi}/3}{4}\tilde{H}^2 - \frac{e^{2\phi/3}}{4}F^2\right)\right] = \frac{3}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left[\sqrt{-g}\left(\frac{8Q_F^2Q_H}{\pi^2}\right)^{2/3}\epsilon_3^2\right] \\ &= 3\left(\frac{8Q_F^2Q_H}{\pi^2}\right)^{2/3}\left(\frac{\delta\epsilon_3^2}{\delta g^{\mu\nu}} - \frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right) = 3\left(\frac{8Q_F^2Q_H}{\pi^2}\right)^{2/3}\left(\epsilon_{3\mu\alpha\beta}\epsilon_{3\nu}{}^{\alpha\beta} - \frac{1}{2}g_{\mu\nu}\right), \end{aligned}$$

where we have used that $\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}$. This is the Einstein equation for the Reissner–Nordström metric with charge $Q = \sqrt{3}\left(\frac{8Q_F^2Q_H}{\pi^2}\right)^{1/3}$. A derivation of the Reissner–Nordström metric starting from the Einstein equation as well as as a description of its properties can be found in section 5 in [27] and section 4.1 in [28]. In the extremal case, i.e. when the mass of the black hole equals its charge, the Reissner–Nordström black hole has one event horizon (in non-extremal case, it has two event horizons).

Near the event horizon, the metric takes the form (see section 1.2 of [29]):

$$ds^2 = -\frac{r^2}{r_0^2}dt^2 + \frac{r_0^2}{r^2}dr^2 + r_0^2 d\Omega_3^2, \quad r_0 = \left(\frac{8Q_H Q_F^2}{\pi^2}\right)^{1/6},$$

where $d\Omega_3$ is the surface element of a 3-sphere S^3. Thus we infer that the area A of the event horizon described by the 3-sphere S^3 is given by $A = 2\pi^2 r_0^3$ and hence the Bekenstein–Hawking entropy is given by:

$$S_{BH} = \frac{A}{4} = 2\pi\sqrt{\frac{Q_H Q_F^2}{2}}. \tag{27}$$

We note that in fact the assumption $\phi_\infty = \phi_h$ could be lifted. Indeed, the near-horizon geometry is unaltered when we change ϕ_∞ adiabatically from the value ϕ_h and hence the result for the entropy still holds for arbitrary values ϕ_h (see discussion in section 2 of [24]).

4.2 The Microscopic Derivation

To derive the Bekenstein–Hawking entropy formula (27) microscopically, we will be counting at weak coupling microscopic configurations of D-branes which at strong coupling produce the black hole considered above. Such states must be BPS as we have been considering an extremal black hole which satisfies a BPS bound, $M = Q$. Additionally, the BPS condition is a requirement for our calculation. Indeed, we already know that BPS branes are stable. Hence, the counting of BPS states at weak coupling will not be affected when changing to strong coupling, which is required such that the counting makes sense for the entropy of the black hole. This is the key idea for the calculation of the entropy.

In fact, we do not count the degeneracy of microscopic configurations of D-branes but instead, we count an index which yields a lower bound on the true degeneracy. The existence of such an index is crucial to derive the entropy formula. Indeed, as the index is chosen to be independent of continuous parameters, it provides us with some freedom during the calculation. In reality, knowing that such an index exists is enough in our calculation. It gives us sufficient freedom to make some crucial approximations but actually the final calculation of the entropy does not rely on the counting of the index but an easier computation involving the Cardy formula.

The D-branes we will be considering also need to carry two non-vanishing charges, since for either $Q_H = 0$ or $Q_F = 0$ the area of the black hole vanishes. This is also sensible intuitively (see section 4 [23] and section 2.5.1 in [25]). Let us consider the example of a string wrapping a circle. Its winding number will correspond to the charge under an electric field of the non-compact world as discussed in section 3.3.4. Now, if the coupling becomes stronger, the wrapped string will produce a black hole without an event horizon. Indeed, the string tension becomes so strong when the coupling changes that it pinches the circle it surrounds, loosing a winding number and hence a horizon. To counter this pinching effect, we can give the string some momentum which will have enough energy to keep the size of the circle finite. We will thus endow a momentum charge and a gauge field charge to the D-branes that we will consider below.

In summary, we consider BPS states with two different charges. Hence, when making the coupling stronger, the black hole produced by these BPS states has the same entropy as the one calculated at weak coupling[7]. Moreover, the black hole inherits the two non-zero charges of the states and thus has a non-vanishing area. At weak coupling, we consider an index giving a lower bound on the microscopic degeneracy of the states. This index gives us enough freedom to make some approximations. We finally compute the entropy with the Cardy formula.

4.2.1 Branes and Charges

We consider D-branes in type IIB superstring theory compactified on $K3 \times S^1$. More precisely we take 1-,3-,5-branes and wrap them around S^1 as well as 0-,2- and 4-cycles of $K3$ respectively, such that they stretch in the S^1 direction. The stretched directions of the branes are string-like. We can then excite left- and right-moving oscillators in the S^1 direction, endowing the states with a momentum charge P. Additionally, these states posses R-R charges. Indeed, as these compactified branes are considered as points in the non-compact world, they couple with the R-R field whose field strength is the 2-form F. We can then write all the R-R charges of the D-branes in a vector Q_F. We can then find a bilinear form encoding the geometry of the system such that contracting Q_F with it yields a number Q_F^2, corresponding to the square of the charge found previously in equation (23). Hence, the black hole produced by those states at strong coupling will have charges characterised by P and Q_F^2. In fact, we can show that $P = Q_H$, yielding exactly the black hole for which we have already computed the Bekenstein–Hawking entropy. Indeed, using the T-duality from type IIB to type IIA the charge P turns into P units of winding around S^1, see our discussion of T-duality in sections 3.1.2 and 3.3.5. Now, we have shown in section 3.3.4 that the winding charge of a brane in the compact direction corresponds to a gauge field charge in the non-compact world. The branes we are considering must couple to a 2-form, which must be $\tilde{H}$ and hence we understand that the winding number is $P = Q_H$.

4.2.2 BPS states

Now that we have identified the correct charges, we need to understand how we get BPS states from the branes in our configuration. As we know from our discussion in section 3.3.3, the D-branes we are considering are actually half-BPS, i.e. they preserve half of the spacetime supersymmetries. However, as our black hole has $\mathcal{N} = 4$ supersymmetry, we must have BPS states which preserve a quarter of the total spacetime supersymmetry. Indeed, we have started with type IIB superstring theory which has 32 spacetime supersymmetries, as discussed in section 3.3.1. Compactifying the theory to five dimensions reduces the number of supersymmetries to 16 while only considering half-BPS states reduces it to 8, we thus need to consider only the states which preserve half of the supersymmetry of the half-BPS

[7] As discussed in section 3.2.3, some states could become degenerate with the BPS states when tuning the coupling which would change the value of the black hole entropy. We conjecture that the occurrences of this phenomenon are negligible in comparison to the total number of states, which is sensible as the correct entropy is found at the end.

D-branes to arrive at 4 supersymmetries.

Since we are interested in counting an index which does not depend on continuous parameters, we can assume that the size of S^1 is much larger than $K3$. The states are then described by a $(1,1)$-sigma model on $S^1 \times \mathbb{R}$ with a target space M. To determine the target space, we can consider the string-like low-energy excitations of the wrapped branes in the S^1 direction. The dynamics of these states are described by a non-linear sigma model whose target space corresponds to M :

$$M = (K3)^k/S_k = \mathrm{Sym}^k(K3), \quad k = \frac{1}{2}Q_F^2 + 1 \tag{28}$$

where S_n denotes the permutation group of n elements (see section 3.1.1 in [26] for an intuitive explanation). In fact, we only need to know that the target space scales as $4(Q_F^2/2+1)$ which is shown in [30] and [31].

Now, we need to restrict to half of the supersymmetries of the sigma model to get the correct BPS states. We use again the fact that we count an index which is independent on the continuous parameters and take the limit where P is fixed and the size of S^1 is taken to infinity. We then get a (4,4)-SCFT in (1+1)-dimensions, as our model becomes clearly scale invariant. The (4,4)-SCFT corresponds to the (2,2)-SCFT discussed in section 3.3.2 with additional structure. The $U(1)$ R-symmetry becomes a $SU(2)$ R-symmetry and there is a global $SU(2) \times SU(2)$ symmetry. We can then consider the RR-sector of the SCFT and restrict the SCFT to the leftmovers only, i.e. in terms of supercharges we set $\overline{L}_0 = \frac{1}{2}(H-P) = 0$ and let $L_0 = \frac{1}{2}(H+P)$ arbitrary. It then follows that $L_0 = L_0 - \overline{L}_0 = P$ and hence L_0 takes integer values. The sigma model is thus described by a subsector of the $(4,4)$-SCFT with the rightmovers unexcited, hence preserving a quarter of the spacetime supersymmetry as desired.

4.2.3 Elliptic Genus

We have shown that the BPS states preserving a quarter of the supersymmetry are described by a SCFT where only leftmovers are excited. We will now identify the index that we are counting and which allowed us to change continuously the parameters of our system.

We have derived in equation (11) the partition function of a generic CFT. Moreover, the (4,4)-SCFT has left- and right-moving sectors with corresponding fermion number F_L, F_R, as seen for (2,2)-SCFT in section 3.3.2. Hence, we can generalize the partition function by adding some chemical potentials $z, \overline{z}$ to each of these conserved charges, such that the partition function takes the form[8]:

$$Z(\tau, q, \overline{q}, y, \overline{y}) = \mathrm{Tr}\left[(-1)^F q^{L_0} q^{\overline{L}_0} y^{F_L} \overline{y}^{F_R}\right],$$

[8]We do not include the central charges from equation (11) for convenience, since they would only yield some constant factors in the partition function. If we would include the central charges, we should then set $\overline{L}_0 = \overline{c}/24$ above instead of $\overline{L}_0 = 0$. We also note that alternatively the factor of $(-1)^F = (-1)^{F_L}(-1)^{F_R}$ could be incorporated in the definitions of y, $\overline{y}$.

where we have defined $F = F_L + F_R$ the total fermion number, $y = e^{2\pi i z}$, $\overline{y} = e^{-2\pi i \overline{z}}$, $q = e^{2\pi i \tau}$, $\overline{q} = e^{2\pi i \overline{\tau}}$. Now, we can specialize the partition function to define the elliptic genus (section 3.2 [32]):

$$\chi(q, , \overline{q}, y) = \mathrm{Tr}\left[(-1)^F y^{F_L} q^{L_0} \overline{q}^{\overline{L}_0}\right]. \tag{29}$$

The factor $(-1)^{F_R}$ will cancel all states for which $\overline{L}_0 > 0$, as for the Witten index discussed in section 3.2.1. We thus infer that the elliptic genus is in fact equivalent to:

$$\chi(q, y) = \mathrm{Tr}\left[(-1)^F y^{F_L} q^{L_0}\right]. \tag{30}$$

Thus, the right-movers do not contribute to this index. Since in our case, the right-movers are annihilated, the elliptic genus exactly counts the BPS states. Of course, as for the Witten index, the value of the elliptic genus does not yield the exact number of BPS states but a lower bound. We recall that the Witten index gives the difference between the number of bosonic and fermionic vacua, from which we could infer a lower bound on the true number of vacua. The elliptic genus is similar in that way as it also has a $(-1)^F$ operator. Moreover, because of this similarity with the Witten index[9], the elliptic genus is also invariant under continuous deformations of the theory (see section 5.2 of [33] or section 2 in [34] for a proof that the elliptic genus is a topological invariant). Hence, all the continuous transformations we have made were justified.

4.2.4 Cardy Formula

We can now apply the Cardy formula from equation (14) to derive the entropy of the black hole[10]. In our case, this formula is indeed applicable as we had calculated the Bekenstein–Hawking formula in a regime where Q_H and Q_F^2 were large, which corresponds to high excitation level of the BPS states.
As M is a (hyperkähler) manifold of dimension $4k := 4\left(Q_F^2/2 + 1\right)$ and as we know from our discussion in section 2.5 and section 2.6 that each bosonic degree of freedom contributes 1 to the central charge and each fermionic partner contributes $\frac{1}{2}$, the central charge is given by $c = 6k$, i.e. $6 = 4 \cdot 1 + 4 \cdot \frac{1}{2}$.
We also know that the excitation level of the leftmovers is given by the eigenvalue of $L_0 = P$ which is in fact equal to Q_H as shown in section 4.2.1. Hence, we can substitute $c = 6\left(Q_F^2/2 + 1\right)$ and $\Delta = Q_H$ in equation (14), which yields the entropy of the black hole:

$$S_{stat} = \ln d(Q_F, Q_H) \sim 2\pi\sqrt{Q_H\left(\frac{1}{2}Q_F^2 + 1\right)}. \tag{31}$$

This formula agrees exactly with the Bekenstein–Hawking entropy previously computed for large Q_F^2 which is exactly the regime in which this calculation was reliable.

[9] We note that for $z = 0$, equation (30) yields the Witten index.

[10] In fact, the Cardy formula was derived for a CFT and not a SCFT as we have here. The correct quantity that should be computed is really the elliptic genus. However, in the limit that we are considering, the Cardy formula yields the correct result. See section 2.4 and 2.5 in [35] for more details regarding computations of the elliptic genus.

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